Rutgers University: Algebra Written Qualifying Exam

January 2019: Problem 2 Solution (Partial)

Exercise. A ring is called completely left reducible if it is a direct sum of left ideals which are simple modules over the ring. For what integers n is the ring $\mathbb{Z}/n\mathbb{Z}$ completely left reducible?

Solution.

Since $\mathbb{Z}/0\mathbb{Z} \cong \mathbb{Z}$ and $\mathbb{Z}/(-n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, we only need to check $n \geq 0$

To be a **simple module:** no nontrivial proper submodules

$n = 1$: $\mathbb{Z}/\mathbb{Z} = \{0\}$ is completely left reducible.

$n = 0$: $\mathbb{Z}$

The ideals of $\mathbb{Z}$ are $\langle m \rangle = m\mathbb{Z}$ for $m \in \mathbb{Z}$.

For $m_1, m_2 \neq 0$,

$$m_1 m_2 \in m_1\mathbb{Z} \cap m_2\mathbb{Z} \text{ and } m_1 m_2 \neq 0$$

$\implies$ $m_1\mathbb{Z}$ and $m_2\mathbb{Z}$ have a nontrivial intersection

$\implies$ $m_1 m_2\mathbb{Z}$ is a submodule of $m_1\mathbb{Z}$ and $m_2\mathbb{Z}$

$m_1\mathbb{Z}$ and $m_2\mathbb{Z}$ are not simple.

Therefore, the only decomposition of $\mathbb{Z}$ is

$$\begin{aligned} \mathbb{Z} &= I_1 \oplus \cdots \oplus I_\ell \\ &= \mathbb{Z} \oplus 0 \oplus \cdots \oplus 0 \end{aligned}$$

Show that $\mathbb{Z}$ is not a simple $\mathbb{Z}-$module.

$\mathbb{Z}$ has nontrivial proper sub-modules (e.g. $2\mathbb{Z}$)

NOT completely left reducible

$n = p$, p prime: Suppose $\mathbb{Z}/p\mathbb{Z}$ is not a simple $\mathbb{Z}/p\mathbb{Z}$ module.

Then it has a nontrivial submodule, which is a subgroup.

But this would have to have order p, so it is not proper

$\implies$ $\mathbb{Z}/p\mathbb{Z}$ is a simple module over itself

$\mathbb{Z}/p\mathbb{Z}$ is completely left reducible

$n = p_1 \dots p_\ell$ where p_i are distinct primes

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\left(p_1 Z\right) \oplus \cdots \oplus \mathbb{Z}/\left(p_\ell \mathbb{Z}\right) \text{ as rings}$$

Where $\mathbb{Z}/\left(p_j\mathbb{Z}\right)$ are ideals.

Show: Each is a simple module over $\mathbb{Z}/n\mathbb{Z}$

(continue solution)